The induction equation

keppens@rijnh.nl

- kinetic 'MHD' models: induction equation in multi-D MHD $+ \nabla \cdot \mathbf{B} = 0$

    $\Rightarrow$ prescribed velocity field, no back-reaction of $\mathbf{B}$ on flow

    $\Rightarrow$ Weiss (1966!) numerical simulations with resistivity

    $\Rightarrow$ expulsion of magnetic flux by eddies

- resistive MHD: Petschek reconnection
- full 2D magnetoconvection simulations (Hurlburt & Toomre 1988)

Weiss kinetic simulations

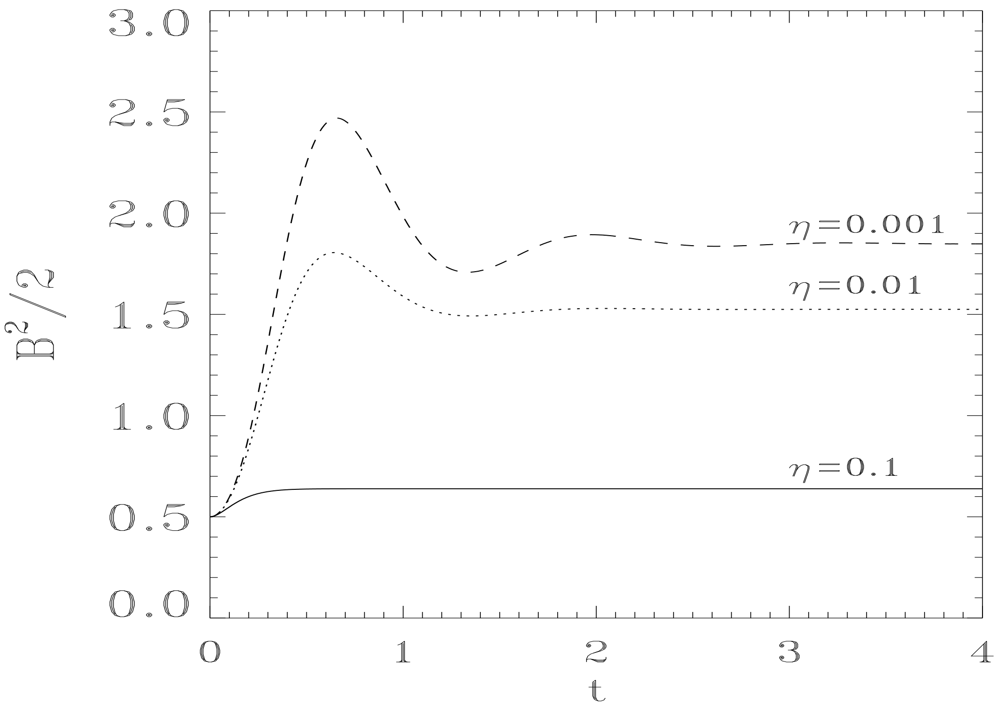
- kinematic modeling: velocity field $\mathbf{v}$ time-invariant
  - $\Rightarrow$ set $\mathbf{v} = [\sin(2\pi x)\sin(2\pi y - \pi/2), \sin(2\pi x + \pi/2)\sin(2\pi y)]$
  - $\Rightarrow$ 2D incompressible flow $\nabla \cdot \mathbf{v} = 0$
  - $\Rightarrow$ models 4 convection cells on unit square $[0,1]^2$
- magnetic field evolution from induction equation
  - $\Rightarrow$ assume $B^2/2 \ll \rho v^2/2$: insignificant magnetic energy
  - $\Rightarrow$ no back-reaction on flow (ignore Lorentz force)
- consider medium with constant resistivity η
  - $\Rightarrow$ induction equation given by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

  - $\Rightarrow$ 2D case: can solve for z-component of vector potential

$$\mathbf{B} = \left(\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, 0 \right)$$

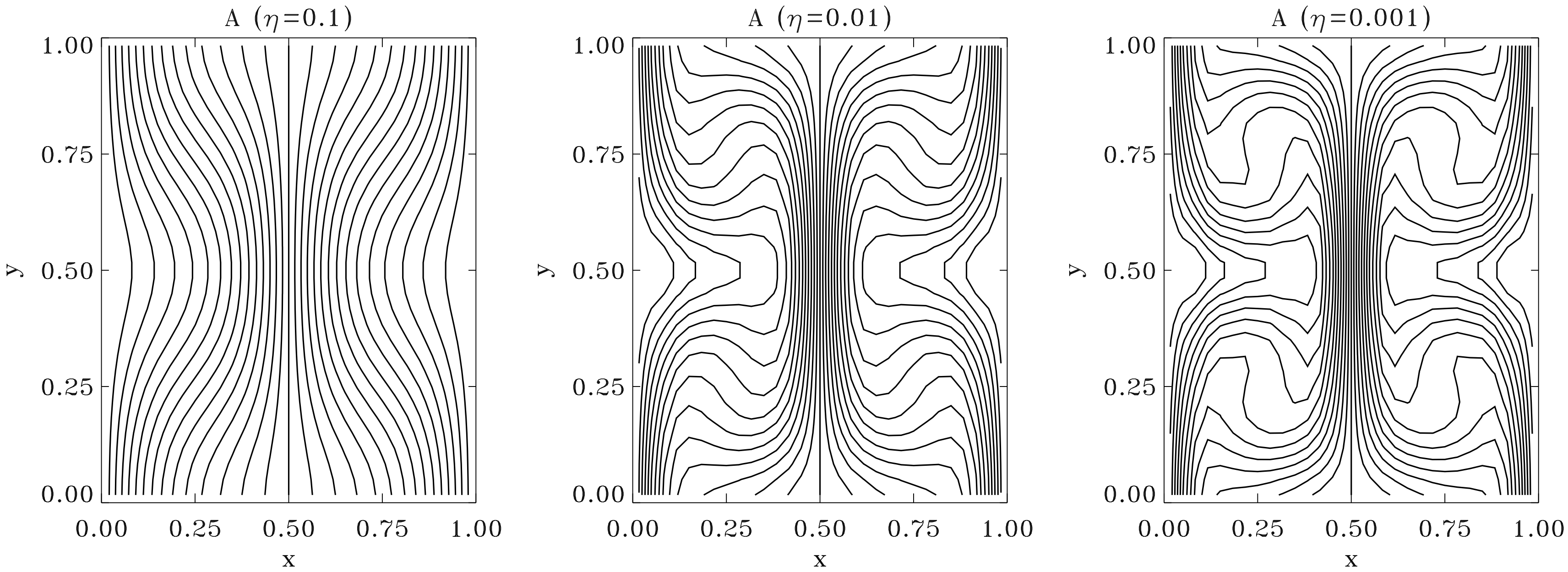
- solve numerically induction equation in 2D, keeping $\nabla \cdot \mathbf{B} = 0$
  - $\Rightarrow$ consider cases $\eta = 0.1$, $\eta = 0.01$, $\eta = 0.001$
  - $\Rightarrow$ from resistive towards ideal MHD case
  - $\Rightarrow$ $\eta = 0$: frozen in limit (numerically: only numerical diffusion)
- perform 30^2 simulations, initial $\mathbf{B} = (0, 1)$
  - $\Rightarrow$ evolution of magnetic energy: 3 phases



- $\Rightarrow$ field amplification, resistive diffusion, steady state
- $\Rightarrow$ turnover: convective term comparable to resistivity

- steady-state configuration:

  $\Rightarrow$ magnetic flux expelled from centre of eddies as $\eta \to 0$

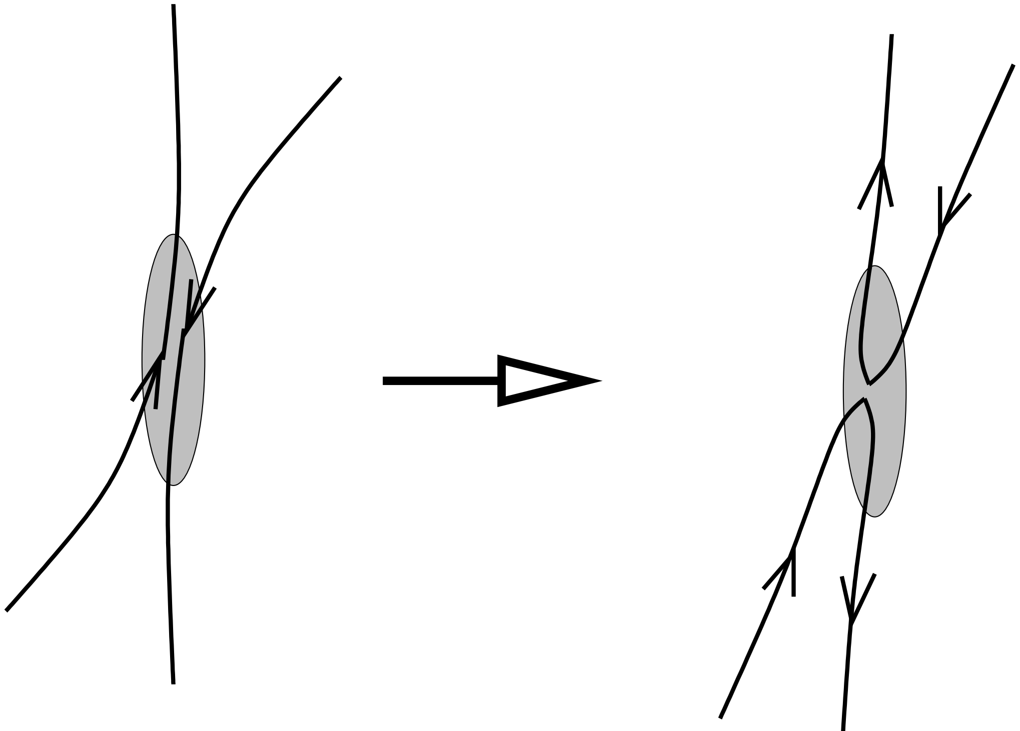


  $\Rightarrow$ flux concentrates at edges of convective cells

  $\Rightarrow$ will return in full 2D MHD magnetoconvection models

Resistive MHD

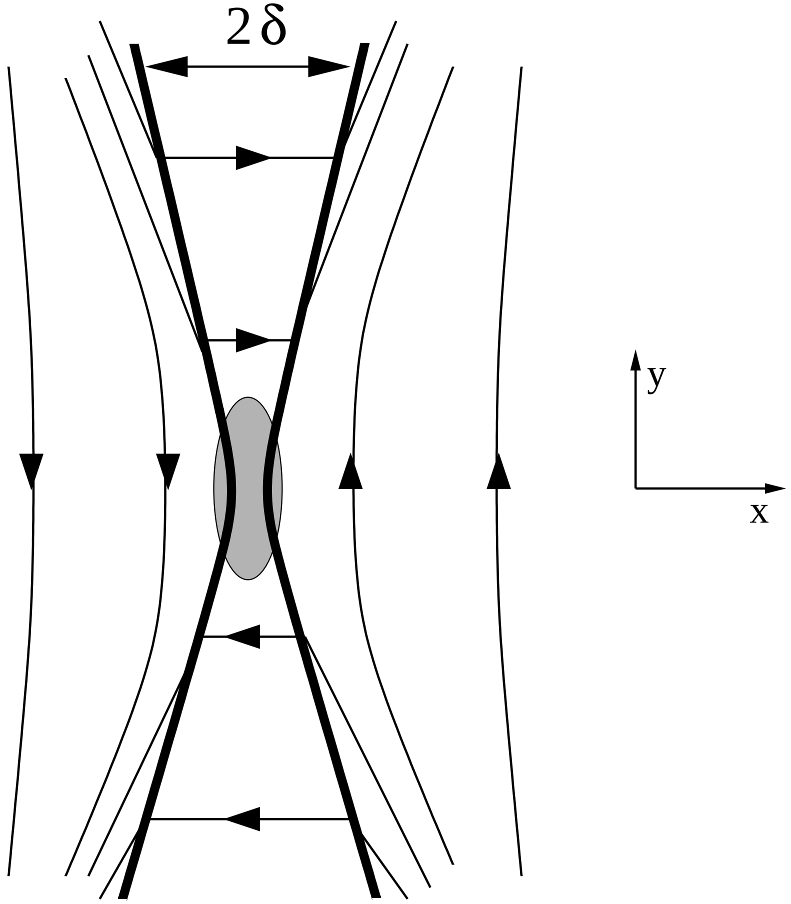
- spatio-temporal resistivity profile $\eta(\mathbf{x}, t)$ introduces
  - $\Rightarrow$ Ohmic heating term in energy equation $S_e = \nabla \cdot (\mathbf{B} \times \eta \mathbf{J})$
  - $\Rightarrow$ diffusion term in induction equation $\mathbf{S}_B = -\nabla \times (\eta \mathbf{J})$
  - $\Rightarrow$ uniform resistivity: $\eta \left(J^2 + \mathbf{B} \cdot \nabla^2 \mathbf{B} \right)$ and $\eta \nabla^2 \mathbf{B}$
- ideal ($\eta = 0$) versus resistive MHD
  - $\Rightarrow$ topological constraint on $\mathbf{B}$ alleviated
  - $\Rightarrow$ field lines can reconnect in regions of strong currents



Petschek reconnection

- Petschek model (1964) for fast magnetic field annihilation
  - $\Rightarrow$ two plasma regions containing oppositely directed field lines
  - $\Rightarrow$ realize steady-state with X-type magnetic neutral point
  - $\Rightarrow$ stationary slow shocks where $\mathbf{B}$ bends towards shock front normal
- at X-point: flow controlled by diffusion
- within region bounded by slow shocks: purely B_x, 'constant' ρ
  - $\Rightarrow$ shock front half-width $\delta(y) = \frac{\rho_e}{\rho_i}\frac{v_{x,e}}{V_{A,e}} \mid y \mid$ (external/internal)
  - $\Rightarrow$ shock fronts have constant opening angle (away from neutral point)
  - $\Rightarrow$ fluid moves to boundary layer and is ejected along it

- stationary configuration



$\Rightarrow$ use symmetry to simulate corner region $[0,1] \times [0,4]$ only

- solve resistive MHD equations incorporating resistivity profile

$$\eta(x,y) = \eta_0 \exp\left[-(x/l_x)^2 - (y/l_y)^2\right]$$

$\Rightarrow$ anomalous η centered on origin $\eta_0 = 0.0001$, $l_x = 0.05$, $l_y = 0.1$

- initial field configuration $\mathbf{B} = (0, \tanh(x/L))$

$\Rightarrow$ initial current sheet width $L = 0.1$

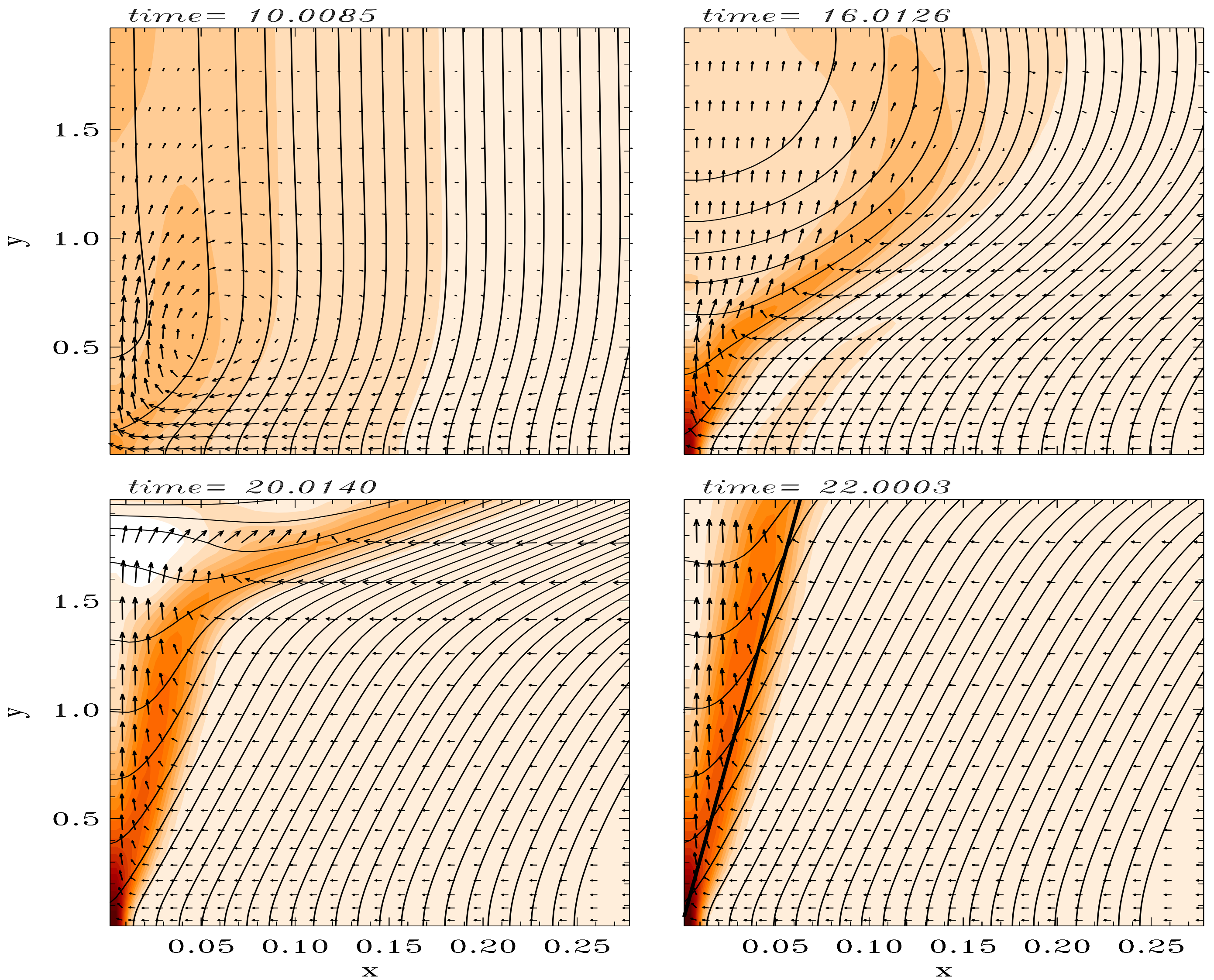
$\Rightarrow$ $\gamma = 5/3$, $p(x) = 1.25 - B_y^2(x)/2$ and $\rho(x) = 2p(x)/\beta_1$

$\Rightarrow$ isothermal initial condition with $\beta_1 = \beta(x=1) = 1.5$

- fix Alfvén Mach number of inflow at $x = 1$: $v_x(x=1) = -0.04$

$\Rightarrow$ evolves self-consistently to Petschek reconnection configuration

- field lines, velocity field, current density evolution



$\Rightarrow$ checks with theoretical opening angle in steady-state!

Magnetoconvection

- full 2D compressible MHD simulations mimicking magneto-convection
  - $\Rightarrow$ Hurlburt & Toomre (1988) numerical MHD simulations
  - $\Rightarrow$ including external gravity, viscosity, resistivity, thermal conduction
  - $\Rightarrow$ periodic sides, impenetrable (to flow) top/bottom boundaries
  - $\Rightarrow$ vertical $\mathbf{B}$ field at top/bottom
  - $\Rightarrow$ fix top temperature and impose bottom temperature gradient
- convectively unstable stratified, uniformly magnetized layer
  - $\Rightarrow$ convective rolls develop
  - $\Rightarrow$ sweep initially uniform vertical field into concentrated flux sheets
  - $\Rightarrow$ studied different configurations of varying $\mathbf{B}$ strength at $t = 0$
  - $\Rightarrow$ transition kinematic to dynamical regime, steady to oscillatory sln.

- $t = 0$ stratification: polytropic relation, linear temperature profile

  $\Rightarrow$ on domain of aspect ratio A: cartesian box $[0, A] \times [0, 1]$

  $\Rightarrow$ linear temperatue $T = T_{\rm top} + 1 - y$, fix $T_{\rm top}$

  $\Rightarrow$ density-pressure polytropic with index m, hence profile

$$\rho(y) = (T_{\rm top} + 1 - y)^m / T_{\rm top}^m$$

- include gravity $\mathbf{g} = -(m+1)\hat{e}_y$ and uniform, vertical $\mathbf{B}$
- gravitational and non-ideal 'source terms' are

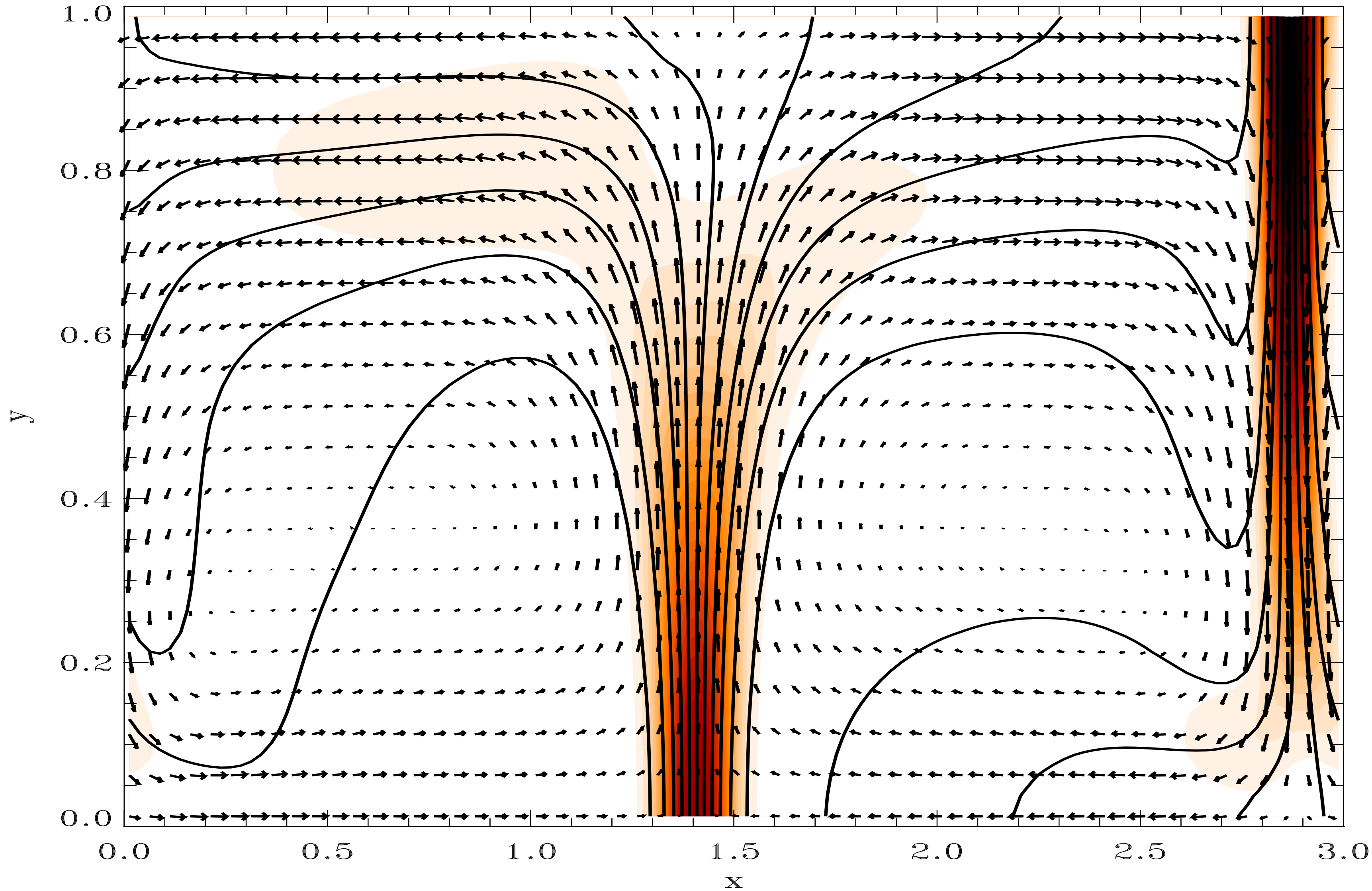
$$\begin{aligned}
\mathbf{S}_{\rho v} &= -\nabla \cdot \left(\nu \hat{\Pi}\right) \quad +\rho \mathbf{g} \\
S_e &= -\nabla \cdot \left(\mathbf{v} \cdot \nu \hat{\Pi}\right) +\rho \mathbf{g} \cdot \mathbf{v} \\
&\qquad +\nabla \cdot (\mathbf{B} \times \eta \mathbf{J}) + \nabla \cdot (\kappa \nabla T) \\
\mathbf{S}_B &= -\nabla \times (\eta \mathbf{J})
\end{aligned}$$

  $\Rightarrow$ viscous terms: traceless part of the total kinetic pressure dyadic

$$\hat{\Pi} = -\left(\nabla \mathbf{v} + \nabla \mathbf{v}^{\rm T}\right) + \frac{2}{3} \hat{I} \left(\nabla \cdot \mathbf{v}\right)$$

- example calculation in 'kinematic' regime (but full 2D MHD!)

  ⇒ field lines, velocity vectors and magnetic pressure



References

- N.O. Weiss, *Proc. Roy. Soc.* **A 293**, 310 (1966)
- H.E. Petschek, *NASA Spec. Publ.* **SP-50**, 425 (1964)
- N.E. Hurlburt and J. Toomre, *Astrophysical Journal* **327**, 920 (1988)
- P.A. Zegeling, R. Keppens, in: **Adaptive Method of Lines**, Ed. Vande Wouwer et al, Chapman & Hall/CRC 2001 (Chapter 4)
- G. Tóth, R. Keppens, M.A. Botchev, *Astron. Astrophys.* **332**, 1159 (1998)
- G. Tóth, *Communications from the Konkoly Observatory of the Hungarian Academy of Science* **12**, 259 (1996)